The Non-linear Relation between Average Size and Perimeter of Lattice Animals

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For $\langle s \rangle_t$, the average number of sites in two dimensional cluster configurations ("lattice animals") with a fixed perimeter t, an asymptotic dependence of the form $\langle s \rangle_t \propto t^{3/2}$ is found. This dependence, for which a scaling theory is still missing, is particularly accurate for the triangular lattice when t is measured by the number of broken bonds.

Owing to their importance for the study of site percolation processes by the perimeter method, doubly-indexed partitions of percolation clusters, g_{st} are well-known (Essam [1], Sykes et al. [2]). One of the indices, say s, denotes the site content of a cluster, whereas the other, say t, denotes the perimeter, i. e. the number of boundary sites needed for its isolation on a lattice. The low density (g_{st} for fixed s) and high density (g_{st} for fixed t) histograms contain all the information needed on the approach to criticality. There is, however, another alternative arrangement of the data, where t denotes the number of boundary bonds, and the recent adaptation of the Potts model to the site percolation problem (Wu [3], Giri et al. [4]) uses this classification. For fixed site content this second partition also corresponds to a strict discrimination of all the clusters by cyclomatic index, via the usual linkage rules, thus adding to their interest. When the perimeter is fixed, however, considerable rearrangement of lattice information is required, since the relation between site perimeter and bond perimeter is highly dependent on the specific form of each cluster. For fixed s the main interest (Stauffer [5], Duarte [6]) lies in the coefficient of the linear dependence of the average $\langle t \rangle = \sum_t t\, g_{st} / \sum_t g_{st}$, which must be assumed valid for both site and bond perimeter. For fixed t, the actual exponent dependence is a problem in itself. From the evolution of the histogram ends, with a range varying from the linear "chain-like" to the high content "compact" clusters the mean site value $\langle s \rangle = \sum_s s\, g_{st} / \sum_s g_{st}$ is subject to the trivial bounds $t^{1.0} < \langle s \rangle < t^{2.0}$. A fitting to a formula of the type $\langle s \rangle \sim t^{\Theta}$ suggests itself. To test this assumption, an extensive range of two dimensional series has been studied. They are given in Duarte [9] and Sykes et al. [2], and the mean value series obtained from them were subject to the usual methods of asymptotic analysis (for a review see Gaunt and Guttmann [7] and Pearce [8]). When the perimeter is measured by the number of broken bonds and the size by the number of occupied sites, the choice will be called a hybrid grouping, whereas in "homogeneous" distributions both the perimeter and the size will be measured by the number of sites. A preliminary study of all groupings by Padé approximants gives support for the formula above, with a singularity located at $\mu = 1$ for $\langle s \rangle \sim \mu^t t^{\Theta}$. For most series too, a singularity at -1 is apparent and its influence is reflected in even-odd oscillations superimposed on ratio method estimates.

Hybrid groupings – their study by the ratio method is highly consistent. For the triangular lattice (up to $t = 42$, Duarte [9]) there is a marked oscillation, of regularly decreasing amplitude around the limiting value, and from successive arithmetic means of the estimates $\Theta = 1.500 \pm 0.002$ is a reasonable final value. For the simple quadratic lattice (up to $t = 22$ Duarte [9]) without any visible oscillation, the estimates are less well-behaved but a value of 1.50 ± 0.02 seems reasonable. This evidence strongly favours 3/2 as the exact value for Θ in two dimensions.

Homogeneous groupings – it is difficult to analyse all the groupings by either method conclusively. For the triangular lattice (Sykes et al. [2]) Table 1 lists estimates $\Theta_t = t(\langle s \rangle_t / \langle s \rangle_{t-1} - 1)$, and alternate linear projections of their envelope. From the last results a pattern seems to be emerging, although the variation of the estimates is still very

Table 1. Ratio analysis of the mean value series for the triangular lattice homogeneous grouping.

t	Θ estimates $\Theta = t(\langle s \rangle_t / \langle s \rangle_{t-1} - 1)$	Alternate linear extrapolants $s_t = (t\,\Theta_t - (t-2)\,\Theta_{t-2})/2$	
14	2.5463	0.655	
15	2.2165	–	1.987
16	2.3304	0.818	–
17	2.1697	–	1.818
18	2.1867	1.037	–
19	2.1157	–	1.657
20	2.1051	1.370	–
21	2.0687	–	1.622
22	2.0528	1.529	–

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Table 2. Ratio analysis of the transformed mean value series for the honeycomb lattice homogeneous grouping.

t	coefficients	Θ estimates	linear extrapolants
3	0.125	—	—
4	0.3125	6.0	—
5	0.53125	3.5	—
6	0.77157	2.714	—
7	1.03064	2.350	—
8	1.60740	2.148	0.734
9	1.60146	2.024	1.031
10	1.91269	1.943	1.216
11	2.24109	1.888	1.340
12	2.58661	1.850	1.426
13	2.94908	1.821	1.481

steep. Consequently an Euler transformation has been performed with the new variable v chosen to be $v = 2\,\mu/(1+\mu)$. The new multiplicative constant is still 1, the singularity at $\mu = -1$ is mapped to infinity and Θ is not affected (Pearce [8]). Unfortunately, the structure of the series is only simultaneously smoothed and improved in convergence for the honeycomb lattice; the results are given in Table 2 for this lattice. From both tables, an estimate of $1.61 \pm .10$, would seem reasonable. The value 1.5 seems unlikely on the basis of the present data. The discrepancy between hybrid and homogeneous groupings could appear attributable to the differing topological definitions of both types of partition. For the latter, non-linear effects in Θ could be the cause of the poor behaviour (Stauffer, private communication).

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